A MATH 1225 Practice Test 4 NAME: SOLUTIONS CRN:________

Multiple Choice

• **No partial credit will be given.** • **Clearly circle one answer.** • **No calculator!**

1. Which of the following **must be true** (you may select more than one answer choice).

   (A) $\displaystyle\int_{-1}^{3} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{3}$

   (B) A function F is called an antiderivative of f on an interval if $F(x) = f'(x)$ for all x in that interval.

   (C) If $G(x) = \displaystyle\int_{3}^{x} t\sin(t)\,dt$, then $G(3) = 0$.

   (D) Let f be a continuous function on $[1, 4]$. If $3 \le f(x) \le 5$, then $\displaystyle\int_{1}^{4} f(x)\,dx > 16$.

   C

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2. Consider the following equation involving f and its first derivative f':

$$f'(x) + f(x) = 0, \text{ for all real numbers } x.$$

   Which of the following **must be true** of such a function f?

   (A) $f(0) = 0$.

   (B) $f(x) = e^x$.

   (C) f is an integrable function.

   (D) There is no such function f.

   C

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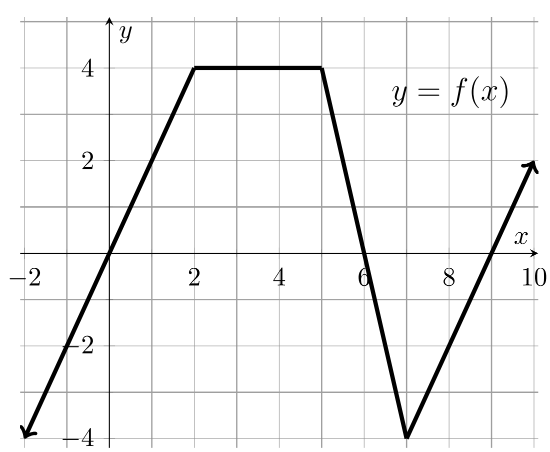
3. The graph of $y = f(x)$ of the function f with domain $(-\infty, \infty)$ is given below.



   Which of the following is the area of the region between the graph of f and the x-axis between $x = 2$ and $x = 9$?

   (A) 14

   (B) 20

   (C) 8

   (D) 6

   B

Free Response

- **Show reasoning that is complete and correct by the standards of this course.**
- **Whenever using theorems, you should explicitly check that all hypotheses are satisfied.**
- **Improper use of (or the absence of) proper notation will be penalized.** • **No calculator!**

4. An experimental and theoretical physicist are studying a particle moving along a fixed line.

   (a) Taking measurements of the particle's velocity (in ft/sec), the experimentalist obtains the following data:

t (sec)	$v(t)$ (ft/sec)
0	0.500
15	0.250
30	0.100
45	0.050
60	0.002

   Approximate the particle's position after 1 minute, provided the particle's initial position is $s = 0$.

   The position after 1 minute (= 60 seconds) is

$$s(60) = \int_0^{60} v(t)\,\mathrm{d}t$$

   We can approximate the integral with a Riemann sum.

   Method 1 (using a left-hand sum):

$$\begin{aligned}\int_0^{60} v(t)\,\mathrm{d}t &\approx 15(0.5 + 0.25 + 0.1 + 0.05)\\ &= 15(0.9) = 13.5\end{aligned}$$

   Method 2 (using a right-hand sum):

$$\begin{aligned}\int_0^{60} v(t)\,\mathrm{d}t &\approx 15(0.25 + 0.1 + 0.05 + 0.002)\\ &= 15(0.429) = 4.02 + 2.01 = 6.03\end{aligned}$$

   (b) Using general principles, the theoretical physicist can write the position of the particle as an integral equation:

$$s(t) = \int_0^{\sqrt{t/15}} \frac{x}{x^4+1}\,\mathrm{d}x$$

   Compute the velocity of the particle at time $t = 1$ minute. Does the predicted velocity agree with the experimentally determined velocity?

   Since $v(t) = \mathrm{d}s/\mathrm{d}t$, we can use the Fundamental Theorem of Calculus to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[s] &= \left(\frac{\left(\sqrt{t/15}\right)}{\left(\sqrt{t/15}\right)^4 + 1}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{t/15}\right]\\ &= \left(\frac{1}{2\left((t/15)^2+1\right)}\right)\left(\frac{1}{15}\right)\end{aligned}$$

   Therefore,

$$v(60) = \left(\frac{1}{2\left((60/15)^2+1\right)}\right)\left(\frac{1}{15}\right) = \frac{1}{2(17)(15)} \approx 0.002$$

   So the experimentally obtained, and the predicted velocity agree.

(c) Using the integral equation from part (b), compute the position of the particle at time $t = 1$ minute. Does the predicted position agree with your approximation from part (a)?

(Note: $\tan^{-1}(2) \approx 1.107$ and $\tan^{-1}(4) \approx 1.326$.)

Since

$$s(60) = \int_0^{\sqrt{60/15}} \frac{x}{x^4+1}\,\mathrm{d}x = \frac{1}{2}\int_0^2 \frac{2x}{(x^2)^2+1}\,\mathrm{d}x$$

we can use $u = x^2$, $\mathrm{d}u = 2x$ as a u-substitution:

$$\begin{aligned}\frac{1}{2}\int_0^2 \frac{2x}{(x^2)^2+1}\,\mathrm{d}x &= \frac{1}{2}\int_0^4 \frac{1}{u^2+1}\,\mathrm{d}u \\ &= \frac{1}{2}\tan^{-1}(u)\Big]_0^4 = \frac{\tan^{-1}(4)}{2} \approx 0.663\end{aligned}$$

This should not agree with any reasonable approximation based on the given data; since $v(t)$ is decreasing, the left-hand sum is an over-estimate and the right-hand sum is an underestimate. However, 0.663 is outside of the range given by our approximations ($6.03 \leq s(60) \leq 13.5$).

5. Compute each of the following integrals:

(a) $\displaystyle\int \frac{u-1}{u^3-u^2}\,\mathrm{d}u$

$$\int \frac{u-1}{u^3-u^2}\,\mathrm{d}u = \int \frac{u-1}{u^2(u-1)}\,\mathrm{d}u = \int \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u} + C$$

(b) $\displaystyle\int_{-1}^{1} (3\,|x| - x)\;\mathrm{d}x$

$$\begin{aligned}\int_{-1}^{1} (3\,|x| - x)\;\mathrm{d}x &= \int_{-1}^{0} (-3x - x)\,\mathrm{d}x + \int_{0}^{1} (3x - x)\,\mathrm{d}x \\ &= \int_{-1}^{0} -4x\,\mathrm{d}x + \int_{0}^{1} 2x\,\mathrm{d}x = \left(-2x^2\right)\Big]_{-1}^{0} + \left(x^2\right)\Big]_{0}^{1} = 2 + 1 = 3\end{aligned}$$

(c) $\displaystyle\int \tan(x) + \csc(x)\,[\csc(x) + 5\cot(x)] + 3^x\,\mathrm{d}x$

$$\begin{aligned}\int \tan(x) + \csc(x)\,[\csc(x) + 5\cot(x)] + 3^x\,\mathrm{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathrm{d}x + \int \csc^2(x)\,\mathrm{d}x + 5\int \csc(x)\cot(x)\,\mathrm{d}x + \int 3^x\,\mathrm{d}x \\ &= -\ln|\cos(x)| - \cot(x) - 5\csc(x) + \frac{3^x}{\ln 3} + C\end{aligned}$$

(d) $\displaystyle\int_{0}^{\pi/4} \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}\,\mathrm{d}x$

Since $\cos(x) > 0$ on the interval $[0, \pi/4]$, we have $\sqrt{\cos^2(x)} = \cos(x)$. Thus,

$$\frac{\cos(x)}{\sqrt{1-\sin^2(x)}} = \frac{\cos(x)}{\sqrt{\cos^2(x)}} = 1$$

Therefore,

$$\int_{0}^{\pi/4} \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}\,\mathrm{d}x = \int_{0}^{\pi/4}\,\mathrm{d}x = \frac{\pi}{4}$$

6. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$ for all x in the interval $(0, 1)$, and $\int_0^1 f(x)\,\mathrm{d}x = 1/3$.

   (a) Sketch the graph of a function f satisfying the properties above.

   One such function is $f(x) = x^2$. Note, $f(0) = 0$, $f(1) = 1$, $f'(x) = 2x > 0$ for all $x > 0$, and

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$$

   The sketch of any function which has the interpolation properties ($f(0) = 0$ and $f(1) = 1$), is increasing, and concave up over part of the subinterval (otherwise the integral would be at least 1/2) would be a reasonable answer.

   (b) What is the graphical relationship (on your sketch from the previous part) between $y = f(x)$ and $x = f^{-1}(y)$?

   They have the same graph in the plane. $y = f(x)$ has x as the independent variable, but $x = f^{-1}(y)$ has y as the independent variable.

   (c) Find the value of the integral $\int_0^1 f^{-1}(y)\,\mathrm{d}y$.

   By part (b), the integral $\int_0^1 f^{-1}(y)\,\mathrm{d}y$ is "complementary" to $\int_0^1 f(x)\,\mathrm{d}x$ within the unit square $[0, 1] \times [0, 1]$ (since they compute the areas of complementary regions in the square). That is,

$$\int_0^1 f^{-1}(y)\,\mathrm{d}y = 1 - \int_0^1 f(x)\,\mathrm{d}x = 1 - \frac{1}{3} = \frac{2}{3}$$

   You intuition may be aided by a picture: refer to the picture you drew in part (a) and your answer from part (b).

7. Suppose that f and g are integrable and that

$$\int_{-2}^4 f(x)\,\mathrm{d}x = 6, \qquad \int_1^4 f(x)\,\mathrm{d}x = 10, \qquad \int_{-2}^4 g(x)\,\mathrm{d}x = 3.$$

   Find the values of the following definite integrals, or indicate that there is not enough information to do so.

   (a) $\displaystyle\int_{-2}^1 3f(x)\,\mathrm{d}x$

$$\int_{-2}^1 3f(x)\,\mathrm{d}x = 3\left(\int_{-2}^1 f(x)\ dx\right) = 3\left(\int_{-2}^4 f(x)\ dx - \int_1^4 f(x)\ dx\right) = 3\,(6 - 10) = 3(-4) = -12$$

   (b) $\displaystyle\int_3^3 g(x)\,\mathrm{d}x$

   $\displaystyle\int_3^3 g(x)\,\mathrm{d}x = 0$ because this is a zero width integral.

   (c) $\displaystyle\int_{-2}^4 f(x)\cdot g(x)\,\mathrm{d}x$

   There is not enough information to determine the value of this integral.